On 2-factors with a bounded number of odd components

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Abstract

A 2-factor in a graph is a spanning 2-regular subgraph, or equivalently a spanning collection of disjoint cycles. In this paper we investigate the existence of 2-factors with a bounded number of odd cycles in a graph. We extend results of Ryjáček, Saito, and Schelp (Closure, 2-factors, and cycle coverings in claw-free graphs, *J. Graph Theory*, **32** (1999), no. 2, 109-117) and show that the number of odd components of a 2-factor in a claw-free graph is stable under Ryjáček's closure operation. We also consider conditions that ensure the existence of of a pair of disjoint 1-factors in a claw-free graph, as the union of such a pair is a 2-factor with no odd cycles.

Keywords: 2-factor, claw-free, closure, disjoint 1-factors

1 Introduction and Notation

Throughout this paper all cycles have an implicit clockwise orientation. For some vertex v on a cycle C we will denote the first, second, and i^{th} predecessor (respectively successor) of v as v^-, v^{--}, and $v^{(-i)}$ (resp. v^+, v^{++}, and $v^{(+i)}$) respectively. Given x and y on C, $C(x,y)$ is the set of vertices $\{x^+, x^{++}, \ldots, y^-\}$ and $C[x,y]$ is the set $\{x, x^+, \ldots, y\}$. We also let xCy (respectively xC^-y) denote the path $xx^+ \ldots y$ (respectively $xx^- \ldots y$). The *circumference* of a graph G, $c(G)$, is the length of a longest cycle in G.

Given a family $\mathcal{F}$ of graphs, a graph G is said to be $\mathcal{F}$*-free* if G contains no member of $\mathcal{F}$ as an induced subgraph. If $\mathcal{F}$ contains a claw, that is $K_{1,3}$, then G is said to be *claw-free*. We denote by $\langle v; x, y, z\rangle$ the claw with central vertex v and leaves x, y, and z; similarly, $\langle v; x_1, x_2, ..., x_t\rangle$ denotes a copy of $K_{1,t}$.

An *r-factor* of a graph G is a spanning r-regular subgraph of G. Thus a *2-factor* of a graph G is a collection of cycles that spans G. A graph is *hamiltonian* if it contains a spanning cycle, which is a 2-factor with exactly one component. The problem of determining when a graph is hamiltonian is a classical and widely studied problem in graph theory (cf. [25, 26]).

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Aside from the hamiltonian problem, there are many results throughout the literature that give conditions ensuring the existence of a 2-factor with given properties. Many results of this type are outlined in [25, 26], and there have been numerous developments since the writing of those surveys.

Let $odd(\mathcal{F})$ denote the number of odd cycles in a 2-factor $\mathcal{F}$ of a graph G and let $odd(G)$ denote the minimum of $odd(\mathcal{F})$ over all 2-factors $\mathcal{F}$ of G. If G does not contain a 2-factor, we will set $odd(G) = \infty$. The focus of this paper is the following.

Problem 1. *Given $k \geq 0$, determine conditions that guarantee a graph G has $odd(G) \leq k$.*

In addition to the general literature on the structure of 2-factors in graphs, $odd(G)$ arises when studying several other interesting problems. For instance, results of Hăggkvist and McGuinness [14] and Huck [19] showed that a cubic bridgeless graph with no vertices of even degree and $odd(G)$ at most four has a family $\mathcal{C}$ of at most five circuits such that each edge of G lies in exactly two members of $\mathcal{C}$. As a consequence, such graphs satisfy the Circuit Double Cover Conjecture (cf. [29, 31]). Understanding the CDC for cubic graphs is crucial given that Seymour [29] noted that a smallest counterexample to the circuit double cover conjecture must be cubic.

Note that $odd(G)$ is a measure of how far G is from having a pair of edge-disjoint perfect matchings, as G contains such a pair if and only if $odd(G) = 0$. Several other related metrics have been considered. Let $B_2(G) = \{(B, B') : B, B' \text{ are edge-disjoint matchings of } G\}$, $P = \{(B, B') \in B_2(G) : |E(B)| + |E(B')| \text{ is maximum}\}$, and H be a maximum matching chosen from the pairs of matchings in P. Then, given a maximum matching M in G, $\mu(G) = |M|/|H|$ measures how close G is to having a pair of maximal disjoint matchings. Note that if G has a perfect matching and $\mu(G) = 1$, then $odd(G) = 0$. Mkrtchyan, Musoyanm, and Tserunyan [22] studied $\mu(G)$ and showed the following.

Theorem 1. *For any graph G,*

$$\mu(G) \leq \frac{5}{4}$$

This bound is tight.

Another measure of how close G is to having a pair of disjoint perfect matchings is given by the ratio of edges covered by a pair of disjoint matchings to the number of vertices in a graph. This notion was explored by Mkrtchyan, Petrosyan, and Vardanyan in [23].

Theorem 2. *Let G be a cubic graph. Then G contains a pair of disjoint matchings covering at least $\frac{4}{5}|V(G)|$ edges in G.*

The problem of determining when a graph has r edge-disjoint perfect matchings for some $r \geq 2$ has also been studied in several contexts. Hilton [15] and Zhang and Zhu [33] examined the number of disjoint 1-factors in regular graphs, and recently Hou [18] considered the problem for nearly-regular graphs (those graphs with $\Delta(G) - \delta(G) \leq 1$). Hoffman and Rodger [16] give necessary and sufficient condition for a complete multipartite graph to contain r edge-disjoint 1-factors.

In this paper, we are generally interested in investigating Problem 1 in the context of forbidden subgraphs. In Section 2 we examine the stability of $odd(G)$ under the Ryjáček closure for claw-free graphs. In Section 3 we specifically consider the case $k = 0$ and examine pairs of forbidden subgraphs that ensure a graph has a pair of disjoint 1-factors. We then present several constructions and open problems related to Problem 1 in highly connected claw-free graphs.

2 Closure and $odd(G)$ for Claw-Free Graphs

In [27] Ryjáček introduced the following closure concept. Call a vertex v *eligible* if $N(v)$ is connected but not complete. The closure of a graph G, denoted $cl(G)$, is constructed by iteratively completing the neighborhood of eligible vertices until none remain.

This closure is an important tool for the study of cycle structure and hamiltonian-type properties in large part due to the following result from [27].

Theorem 3. *Let G be a claw-free graph. Then*

1. *the closure $cl(G)$ is well-defined,*
2. *there is a triangle-free graph H such $cl(G) = L(H)$, where $L(H)$ is the line graph of H,*
3. *$c(G) = c(cl(G))$.*

Given a graph class $\mathcal{C}$, we say that a property π is *stable* in $\mathcal{C}$ provided that every graph G in $\mathcal{C}$ has property π if and only if $cl(G)$ has property π. Thus, Theorem 3 implies that hamiltonicity (and more generally the property "$c(G) = k$") is stable in the class of claw-free graphs.

Of particular interest here, Ryjáček, Saito, and Schelp [28] showed the following.

Theorem 4. *If G is a claw-free graph, then G has a 2-factor with at most k cycles if and only if $cl(G)$ has a 2-factor with at most k cycles.*

The main result of this section is the following extension of Theorem 4.

Theorem 5. *If G is a claw-free graph, then $odd(G) \leq k$ if and only if $odd(cl(G)) \leq k$.*

Proof. Clearly, if $odd(G) \leq k$, then $odd(cl(G)) \leq k$, so we proceed by demonstrating the converse. Let G be a claw-free graph and let $G = G_0, G_1, \ldots, G_t = cl(G)$ be a sequence of graphs such that G_i is the graph formed by local completion of G_{i-1} at vertex v_i. Suppose to the contrary that $odd(G_t) \leq k$ but $odd(G) > k$. Let $i \geq 1$ be the minimum integer such that G_i has a 2-factor $\mathcal{C}$ with at most k odd cycles and denote v_i as v. Let $E^i_{\mathcal{C}} = (E(\mathcal{C}) \cap E(G_i)) \setminus E(G_{i-1})$, so that $E^i_{\mathcal{C}}$ is the set of edges of the 2-factor $\mathcal{C}$ in G_i which are not in G_{i-1}, and observe that the endpoints of these edges are all in the neighborhood of v. Choose $\mathcal{C}$ to minimize $|E^i_{\mathcal{C}}|$ and furthermore, among all such choices of $\mathcal{C}$ select an edge $e \in E^i_{\mathcal{C}}$ such that $dist_{\mathcal{C}}(e, v)$ is maximum. For each cycle in $\mathcal{C}$, let C^w denote the cycle containing a particular vertex w and similarly let C^f denote the cycle containing a particular edge f. Let $e = u_1u_2$, where u_2 is the predecessor of u_1 in C^e. Let P be a shortest path between u_1 and u_2 in G_{i-1}. Since G_{i-1} is claw-free, P has length at most four; denote the vertices of P as $u_1 = x_1x_2...x_\ell = u_2$ with $3 \leq \ell \leq 4$. We begin by noting certain edges and non-edges of G_{i-1}.

Claim 1. *If $C^v = C^e$, the following hold:*

1. *$u_2v^- \notin E(G_{i-1})$,*
2. *$u_1v^+ \notin E(G_{i-1})$,*
3. *$v^-v^+ \notin E(G_{i-1})$,*

4. if $u_2 \neq v^+$, then $u_2v^+ \in E(G_{i-1})$.

Proof of Claim. If u_2v^- is an edge, then replacing C^v with $u_2v^-C^{v-}u_1vC^vu_2$ results in a 2-factor $\mathcal{C}'$ such that $odd(\mathcal{C}') \leq k$ and $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$, a contradiction. Similarly, if $u_1v^+ \in E(G_{i-1})$, replacing C^v with $u_1v^+C^vu_2vC^{v-}$ contradicts the minimality of $|E^i_{\mathcal{C}}|$, also a contradiction.

Suppose C^v is a triangle, then $v^- = u_1$ and $v^+ = u_2$, so clearly, v^-v^+ is not an edge of G_{i-1}. If C^v is a four cycle, then $u_1 = v^-$ or $u_2 = v^+$, and in either case, from above, $v^-v^+ \notin E(G_{i-1})$. Lastly if C^v has more than four vertices and $v^-v^+ \in E(G_{i-1})$, then replacing C^v with $v^-v^+C^vu_2vu_1C^vv^-$ contradicts the minimality of $|E^i_{\mathcal{C}}|$.

Finally, consider $\langle v; u_1, u_2, v^+\rangle$. By assumption, $u_1u_2 \notin E(G_{i-1})$, and since part (2) holds $u_1v^+ \notin E(G_{i-1})$, thus since G_{i-1} is claw-free, $u_2v^+ \in E(G_{i-1})$. □

The following will also be useful throughout the remainder of the proof.

Claim 2. *Let $f = w_1w_2 \in \mathcal{C}$ such that $C^e \neq C^f$. If $u_1w_1 \in E(G_{i-1})$ and $u_2w_2 \in E(G_{i-1})$, then there is a cycle C such that $V(C) = V(C^e) \cup V(C^f)$ where $e \notin C$.*

Proof of Claim. Specifically assume that w_1 is the predecessor of w_2 (without loss of generality) on C^f, and replace C^e and C^f with $w_1u_1C^{u_1}u_2w_2C^{w_2}w_1$, resulting in a 2-factor $\mathcal{C}'$ such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. □

We now proceed by considering $dist_{\mathcal{C}}(e, v)$.

Case 1: $dist_{\mathcal{C}}(e, v) \geq 3$.

Suppose $C^v \neq C^e$, and consider v^+, the successor of v on C^v. The claw $\langle v; v^+, u_1, u_2\rangle$ in G_{i-1} implies that without loss of generality v^+u_2 is an edge in G_{i-1}, since $e = u_1u_2 \notin E(G_{i-1})$. However, Claim 1 implies that $v^+u_2 \notin E(G_{i-1})$, a contradiction.

Thus $C^v = C^e$. Denote the vertices of C^v in order as $v, v_1, v_2, \ldots, v_r$ so that $\{u_1, u_2\} \subset \{v_3, v_4, \ldots, v_{r-2}\}$. By Claim 1, $u_2v^- \notin E(G_{i-1})$, $u_1v^+ \notin E(G_{i-1})$, and $u_2v^+ \in E(G_{i-1})$. Considering the claw $\langle v; u_1, u_2, v^-\rangle$ we see that $u_1v^- \in E(G_{i-1})$. When C^v is an odd cycle, replacing C^v with $u_1C^vv^-u_1$ and vC^vu_2v, results in a 2-factor $\mathcal{C}'$ with $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. If C^v is an even cycle, then replacing C^v with $u_1C^vv^-u_1$ and vC^vu_2v will result in two even cycles or two odd cycles. When this results in two even cycles this 2-factor $\mathcal{C}'$ has $odd(\mathcal{C}') \leq k$ and $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. When the cycles $u_1C^vv^-u_1$ and vC^vu_2v are both odd cycles, then instead replacing C^v with $u_1C^vvu_1$ and $v^+C^vu_2v^+$ results in two even cycles to contradict the minimality of $|E^i_{\mathcal{C}}|$. This completes Case 1.

Case 2: $dist_{\mathcal{C}}(e, v) = 2$.

Without loss of generality, assume $e = v^{--}v^{(-3)} = u_1u_2$. Note that since $dist_{\mathcal{C}}(e, v) = 2$, Claim 1 implies $v^+v^{(-3)} \in E(G_{i-1})$.

We also claim that if $x_2 \neq v^-$, then $C^{x_2} \neq C^v$. Indeed, note that if x_2^-v or x_2^+v is an edge of G_{i-1}, then replacing C^v with either $x_2^-vv^-v^{--}x_2C^vv^{(-3)}v^+C^vx_2^-$ or $x_2^+vv^-v^{--}x_2C^{v-}v^+v^{(-3)}C^{v-}x_2^+$ results in a 2-factor $\mathcal{C}'$ such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. Also, this implies that $x_2^- \neq v^+$ and $x_2^+ \neq v^{(-3)}$ since v cannot be adjacent to x_2^+ or x_2^-. Thus, since $\langle x_2; v, x_2^-, x_2^+\rangle$ is not induced, $x_2^-x_2^+ \in E(G_{i-1})$. However, exchanging C^v with $vv^-v^{--}x_2v$ and $v^+C^vx_2^-x_2^+C^vv^{(-3)}v^+$ contradicts the minimality of $|E^i_{\mathcal{C}}|$. Therefore $x_2 \notin V(C^v)$ when $x_2 \neq v^-$.

Case 2.1: $\ell = 3$, *so that in particular* $P = v^{--}x_2v^{(-3)} = u_1x_2u_2$.

By Claim 1, $u_2 = v^{(-3)}$ is not adjacent to v^-, so $x_2 \neq v^-$, and further $C^{x_2} \neq C^v$.

If $x_2^+ v^{(-3)} \in E(G_{i-1})$ or $x_2^+ v^{--} \in E(G_{i-1})$, then replacing C^v and C^{x_2} with either $x_2^+ C^{x_2} x_2 v^{--} C^v v^{(-3)} x_2^+$ or $x_2^+ v^{--} C^v v^{(-3)} x_2 C^{x_2 -} x_2^+$ results in a 2-factor $\mathcal{C}'$ with at most k odd cycles such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. Thus, since $\langle x_2; x_2^+, v^{(-3)}, v^{--}\rangle$ is not induced in G_{i-1}, P contains two internal vertices.

Case 2.2: $C^{x_2} = C^v$.

At the start of Case 2, we showed that if $x_2 \neq v^-$, then $C^v \neq C^{x_2}$, so we can assume $x_2 = v^-$ and $\ell = 4$. By Claim 1, $x_3 \neq v^+$, since $v^- v^+ \notin E(G_{i-1})$. Consider $\langle x_3; v^{(-3)}, v^-, x_3^- \rangle$. If $v^{(-3)} x_3^- \in E(G_{i-1})$, we may replace C^v with $v^{(-3)} x_3^- C^{v-} v v^{--} v^- x_3 C^v v^{(-3)}$ resulting in a 2-factor $\mathcal{C}'$ with $odd(\mathcal{C}') \leq k$ such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. If $v^- x_3^- \in E(G_{i-1})$, replacing C^v with $v^- x_3^- C^{v-} v^+ v^{(-3)} C^{v-} x_3 v v^{--} v^-$ similarly contradicts the minimality of $|E^i_{\mathcal{C}}|$. However, from Claim 1, $v^- v^{(-3)} \notin E(G_{i-1})$. Thus we may assume that x_3 is not on C^v, specifically $C^{x_3} \neq C^v$.

Consider the claw $\langle x_3; x_3^+, v^-, v^{(-3)} \rangle$. Since $v^- v^{(-3)}$ is not an edge, one of $x_3^+ v^-$ or $x_3^+ v^{(-3)}$ is an edge in G_{i-1}. If $x_3^+ v^- \in E(G_{i-1})$, replace C^v and C^{x_3} with $v v^{--} v^- x_3^+ C^{x_3} x_3 v^{(-3)} C^{v-} v$. Otherwise, if $x_3^+ v^{(-3)} \in E(G_{i-1})$, replace C^v with $v v^{--} v^- x_3 C^{x_3 -} x_3^+ v^{(-3)} C^{v-} v$. Both scenarios result in a 2-factor $\mathcal{C}'$ with at most k odd cycles such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$, contradicting the minimality of $|E^i_{\mathcal{C}}|$. Therefore x_2 is not v^-, and this completes Case 2.2.

Case 2.3: $C^{x_3} = C^v$.

Suppose $x_3 = v^+$. If $v^{--} x_2^- \in E(G_{i-1})$, then replacing C^v and C^{x_2} with $x_2 v^+ C^v v^{(-3)} v v^- v^{--} x_2^- C^{x_2 -} x_2$ contradicts the minimality of $|E^i_{\mathcal{C}}|$. If $v^+ x_2^- \in E(G_{i-1})$, then by replacing C^v and C^{x_2} with $v^+ x_2^- C^{x_2 -} x_2 v^{--} v^- v v^{(-3)} C^{v-} v^+$ in $\mathcal{C}$ we contradict the minimality of $|E^i_{\mathcal{C}}|$. As Claim 1 shows, $v^{--} v^+ \notin E(G_{i-1})$, so we have that $\langle x_2; v^{--}, v^+, x_2^- \rangle$ is induced, thus $x_3 \neq v^+$.

By Claim 1, $v^- v^{(-3)} \notin E(G_{i-1})$. If $x_2 v^{(-3)} \in E(G_{i-1})$, then $v^{--} x_2 v^{(-3)}$ contradicts the minimality of P. Thus, since $\langle v; v^-, v^{(-3)}, x_2 \rangle$ is not induced, $x_2 v^- \in E(G_{i-1})$. If $v^- x_2^- \in E^i_{\mathcal{C}}$, then replacing C^v and C^{x_2} with $v^- x_2^- C^{x_2 -} x_2 v C^v v^-$ results in a 2-factor $\mathcal{C}'$ with $odd(\mathcal{C}') \leq k$ such that $dist_{\mathcal{C}'}(e, v) > dist_{\mathcal{C}}(e, v)$, a contradiction to the maximality of $dist_{\mathcal{C}}(e, v)$.

Since $\langle x_2; v^-, x_2^-, x_3 \rangle$ is not induced, either $x_3 x_2^-$ or $v^- x_3$ is an edge in G_{i-1}. First, assume that $x_3 x_2^- \in E(G_{i-1})$. If $x_3^+ x_2^-$ or $x_3^- x_2^-$ is an edge in G_{i-1}, then either $x_3^+ x_2^- C^{x_2 -} x_2 v^{--} C^v x_3 v^{(-3)} C^{v-} x_3^+$ or $x_3^- C^{v-} v^+ v^{(-3)} C^{v-} x_3 v v^- v^{--} x_2 C^{x_2} x_2^- x_3^-$ can replace C^v and C^{x_2}, resulting in a 2-factor $\mathcal{C}'$ with at most k odd cycles such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. Since $\langle x_3; x_3^+, x_3^-, x_2^- \rangle$ is not induced, $x_3^- x_3^+ \in E(G_{i-1})$. But then replacing C^v and C^{x_2} with $x_2^- C^{x_2 -} x_2 v^{--} C^v x_3^- x_3^+ C^v v^{(-3)} x_3 x_2^-$ contradicts the minimality of $|E^i_{\mathcal{C}}|$. This means that $v^- x_3 \in E(G_{i-1})$, so since we could have chosen $P = v^{--} v^- x_3 v^{(-3)}$, from Case 2.2, the minimality of $|E^i_{\mathcal{C}}|$ is contradicted, completing Case 2.3.

Case 2.4: $C^{x_3} = C^{x_2}$.

By Claim 2, $x_3 \notin \{x_2^+, x_2^-\}$. Suppose C^{x_2} is a 4-cycle, and consider the claw $\langle x_2; x_2^+, x_2^-, v^{--} \rangle$. If $x_2^+ x_2^- \in E(G_{i-1})$, then $v^{--} C^v v^{(-3)} x_3 x_2^- x_2^+ x_2 v^{--}$ may replace C^v and C^{x_2}, resulting in a 2-factor $\mathcal{C}'$ with at most k odd cycles such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$, a contradiction. If $x_2^+ v^{--}$ (respectively $x_2^- v^{--}$) is an edge in G_{i-1}, then replace C^v and C^{x_2} with $v^{--} C^v v^{(-3)} x_3 C^{x_2} x_2^+ v^{--}$ (respectively $v^{--} C^v v^{(-3)} x_3 C^{x_2 -} x_2^- v^{--}$) to contradict the minimality of $|E^i_{\mathcal{C}}|$. Thus, we can assume C^{x_2} is not a 4-cycle.

If both $x_2^- x_2^+$ and $x_3^- x_3^+$ are edges in G_{i-1}, then replacing C^v and C^{x_2} with $v^{--} C^v v^{(-3)} x_3 x_2 v^{--}$ and $x_2^+ C^{x_2} x_3^- x_3^+ C^{x_2} x_2^- x_2^+$ yields a 2-factor $\mathcal{C}'$ with $odd(\mathcal{C}') \leq k$ and $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$.

First assume $x_2^- x_2^+ \notin E(G_{i-1})$. Since $\langle x_2; x_2^+, x_2^-, v \rangle$, is not induced, one of $x_2^- v$ or $x_2^+ v$ is an edge of G_{i-1}. Assume without loss of generality that $x_2^- v \in E(G_{i-1})$. Consider the claw $\langle v; x_2^-, v^-, v^+\rangle$. By Claim 1, $v^-v^+ \notin E(G_{i-1})$. If $v^+x_2^- \in E(G_{i-1})$, then replacing C^v and C^{x_2} with $vv^{(-3)}C^{v-}v^+x_2^-C^{x_2-}x_2v^{--}v^-v$ results in a 2-factor $\mathcal{C}'$ with $odd(\mathcal{C}') \leq k$ such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. If $v^-x_2^- \in E(G_{i-1})$, then replace C^v and C^{x_2} with $vC^vv^{(-3)}v$ and $x_2C^{x_2}x_2^-v^-v^{--}x_2$ to again contradict the minimality of $|E^i_{\mathcal{C}}|$. However G_{i-1} is claw-free, implying that $x_2^-x_2^+ \in E(G_{i-1})$, so $x_3^-x_3^+ \notin E(G_{i-1})$.

Since $x_3^+x_3^- \notin E(G_{i-1})$ and $\langle x_3; x_3^+, x_3^-, v\rangle$ is not induced, one of x_3^+v or x_3^-v is an edge in G_{i-1}. Without loss of generality, assume that $x_3^+v \in E(G_{i-1})$. Again, by Claim 1 $v^-v^+ \notin E(G_{i-1})$. Thus, since $\langle v; x_3^+, v^-, v^+\rangle$ is not induced, either $x_3^+v^- \in E(G_{i-1})$ or $x_3^+v^+ \in E(G_{i-1})$. If $x_3^+v^- \in E(G_{i-1})$, then $v^-x_3^+C^{x_2}x^3v^{(-3)}C^{v-}vv^{--}v^-$ can replace C^v and C^{x_2} to contradict the minimality of $|E^i_{\mathcal{C}}|$. Otherwise, if $x_3^+v^+ \in E(G_{i-1})$, replacing C^v and C^{x_2} with $x_2v^{--}v^-vx_2$ and $v^+C^vv^{(-3)}x_3C^{x_2-}x_2^+x_2^-C^{x_2-}x_3^+v^+$ contradicts the minimality of $|E^i_{\mathcal{C}}|$. This contradiction completes Case 2.4.

Case 2.5: C^{x_3}, C^{x_2}, *and* C^v *are distinct.*

Since $\langle x_2; x_3, x_2^+, v^{--}\rangle$ cannot be induced, x_3v^{--}, $x_2^+v^{--}$, or $x_3x_2^+$ is an edge of G_{i-1}. Note that $x_3v^{--} \notin E(G_{i-1})$, as otherwise $v^{--}x_3v^{(-3)}$ would contradict the minimality of P. Suppose $x_2^+v^{--} \in E(G_{i-1})$, and consider $\langle v; x_2, v^-, v^{(-3)}\rangle$. If $x_2v^{(-3)} \in E(G_{i-1})$, then $v^{--}x_2v^{(-3)}$ contradicts the minimality of P. Otherwise, if $v^-v^{(-3)} \in E(G_{i-1})$, then replacing C^v with $v^{--}v^-v^{-3}C^{v-}vv^{--}$ yields a 2-factor C' with at most k odd cycles such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. This means $x_2v^- \in E(G_{i-1})$. However, now replace C^v and C^{x_2} with $v^{--}x_2^+C^{x_2}x_2v^-v^{--}$ and $vC^vv^{(-3)}v$ to contradict the minimality of $|E^i_{\mathcal{C}}|$. Therefore $x_3x_2^+ \in E(G_{i-1})$.

Consider $\langle x_3; x_3^-, x_2^+, v^{(-3)}\rangle$. This cannot be an induced claw, so at least one of $x_3^-x_2^+$, $x_2^+v^{(-3)}$, or $x_3^-v^{(-3)}$ must be an edge of G_{i-1}. When $x_3^-x_2^+ \in E(G_{i-1})$, replace C^v, C^{x_2}, and C^{x_3} with $x_2^+x_3^-C^{x_3-}x_3v^{(-3)}C^{v-}v^{--}x_2C^{x_2-}x_2^+$, resulting in a 2-factor $\mathcal{C}'$ such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. If $x_2^+v^{(-3)} \in E(G_{i-1})$, replace C^v and C^{x_2} with $x_2^+v^{(-3)}C^{v-}v^{--}x_2C^{x_2-}x_2^+$ to contradict the minimality of $|E^i_{\mathcal{C}}|$. Finally, the edge $x_3^-v^{(-3)}$ allows us to replace C^v, C^{x_2}, and C^{x_3} with $x_3^-v^{(-3)}C^{v-}v^{--}x_2C^{x_2-}x_x^+x_3C^{x_3}x_3^-$, again contradicting the minimality of $|E^i_{\mathcal{C}}|$. This contradiction completes the proof of Case 2.5, and further Case 2 is complete.

Case 3: $dist_{\mathcal{C}}(e, v) = 1$.

Without loss of generality, assume $e = v^-v^{--}$. By Claim 1, $v^-v^+ \notin E(G_{i-1})$, so $x_2 \neq v^+$.

Case 3.1: $C^{x_2} = C^v$.

If $x_2 = v^{++}$, then replacing C^v with $v^-x_2C^vv^{--}v^+vv^-$ results in a 2-factor $\mathcal{C}'$ with at most k odd cycles such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$, contradicting the minimality of $|E^i_{\mathcal{C}}|$. Thus $dist_{C^v}(v, x_2) \geq 3$. If $x_2^-x_2^+ \in E(G_{i-1})$, then replacing C^v with $vx_2v^-C^{v-}x_2^+x_2^-C^{v-}v$ results in a 2-factor $\mathcal{C}'$ such that $dist_{\mathcal{C}'}(e, v) > dist_{\mathcal{C}}(e, v)$. If $x_2^-v^-$ or $x_2^+v^-$ is an edge of G_{i-1} then replacing C^v with $vC^vx_2^-v^-x_2C^vv^{--}v$ or $vC^vx_2v^-x_2^+C^vv^{--}v$ contradicts the minimality of $|E^i_{\mathcal{C}}|$. Thus $\langle x_2; x_2^-x_2^+, v^-\rangle$ is induced, a contradiction. Therefore $x_2 \notin V(C^v)$.

Case 3.2: $\ell = 3$, *so in particular* $P = v^-x_2v^{--} = u_1x_2u_2$.

By Claim 2, $\langle x_2; x_2^+, v^{--}, v^-\rangle$ is induced in G_{i-1}, so P must have two internal vertices.

Case 3.3: $C^{x_3} = C^v$.

First assume $x_3 = v^+$. If $v^+x_2^- \in E(G_{i-1})$, then we can replace C^v and C^{x_2} with $v^+x_2^-C^{x_2-}x_2v^-vv^{--}C^{v-}v^+$ to contradict the minimality of $|E^i_{\mathcal{C}}|$. If $v^-x_2^- \in E(G_{i-1})$ then replacing C^v and C^{x_2} with $v^-x_2^-C^{x_2-}x_2v^+C^vv^{--}vv^-$

contradicts the minimality of $|E^i_{\mathcal{C}}|$. By Claim 1 $v^-v^+ \notin E(G_{i-1})$, so $\langle x_2; x_2^-, v^+, v^- \rangle$ is induced in G_{i-1}, a contradiction implying $x_3 \neq v^+$.

If $x_2^- v^- \in E(G_{i-1})$, then replace C^v and C^{x_2} with $vx_2C^{x_2}x_2^-v^-v^{--}C^{v^-}v$ to contradict the maximality of $dist_{\mathcal{C}}(e,v)$. Since $x_3v^- \in E(G_{i-1})$, the claw $\langle x_2; v^-, x_3, x_2^- \rangle$ is induced unless $x_2^-x_3 \in E(G_{i-1})$.

If $x_2^-v^{--} \in E(G_{i-1})$ replacing C^v and C^{x_2} with $x_2^-C^{x_2^-}x_2v^-C^vv^{--}x_2^-$ results in a 2-factor $\mathcal{C}'$ with $odd(\mathcal{C}') \leq k$ such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. If $x_3^-v^{--} \in E(G_{i-1})$, then replacing C^v and C^{x_2} with $v^{--}C^{v^-}x_3x_2^-C^{x_2^-}x_2v^-C^vx_3^-v^{--}$ similarly contradicts the minimality of $|E^i_{\mathcal{C}}|$. Finally if $x_2^-x_3^- \in E(G_{i-1})$, then $vC^vx_3^-x_2^-C^{x_2^-}x_2v^-C^{v^-}x_3v$ can replace C^v and C^{x_2} to contradict the maximality of $dist_{\mathcal{C}}(e,v)$. Since $\langle x_3; x_3^-, v^{--}, x_2^- \rangle$ cannot be induced, $x_3 \notin C^v$.

Case 3.4: $C^{x_3} = C^{x_2}$.

By Claim 2, $x_3 \neq x_2^-$ and $x_3 \neq x_2^+$. Without loss of generality, $v^-x_2^+ \notin E(G_{i-1})$ (respectively $v^-x_2^-$), as otherwise, replacing C^v and C^{x_2} with $v^-x_2^+C^{x_2}x_2vC^vv^-$ results in a 2-factor such that $C^v = C^{x_2}$, which was considered in Case 3.1. Thus, considering $\langle x_2; x_2^-, x_2^+, v^- \rangle$, $x_2^-x_2^+ \in E(G_{i-1})$.

Now, consider the claw $\langle x_3; x_3^+, x_2, v^{--} \rangle$. By the minimality of P, $x_2v^{--} \notin E(G_{i-1})$. If $v^{--}x_3^+ \in E(G_{i-1})$, then replacing C^v and C^{x_2} with $v^{--}x_3^+C^{x_2}x_2^-x_2^+C^{x_2}x_3x_2v^-C^vv^{--}$ results in a 2-factor $\mathcal{C}'$ with at most k odd cycles such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. Otherwise, if $x_2x_3^+ \in E(G_{i-1})$, then replace C^v and C^{x_2} with $x_2x_3^+C^{x_2}x_2^-x_2^+C^{x_2}x_3v^{--}C^{v^-}v^-x_2$ to contradict the minimality of $|E^i_{\mathcal{C}}|$. This contradiction completes the proof of Case 3.4.

Case 3.5: C^{x_3}, C^{x_2} *and* C^v *are distinct.*

First, consider the claw $\langle x_2; x_2^+, v^-, x_3 \rangle$. As in Case 3.4, $v^-x_2^+ \notin E(G_{i-1})$ as otherwise, replacing C^v and C^{x_2} with $v^-x_2^+C^{x_2}x_2vC^vv^-$ results in a 2-factor such that $C^v = C^{x_2}$, considered in Case 3.1. By the minimality of P, $v^-x_3 \notin E(G_{i-1})$. So we have that $x_3x_2^+ \in E(G_{i-1})$.

Next, by the minimality of P, $x_2v^{--} \notin E(G_{i-1})$. If $x_3^+x_2 \in E(G_{i-1})$, replacing C^{x_2} and C^{x_3} with $x_2x_3^+C^{x_3}x_3x - 2^+C^{x_2}x_2$ yields a 2-factor such that $C^{x_2} = C^{x_3}$, which is Case 3.4. Thus, since $\langle x_3; x_2, x_3^+, v^{--} \rangle$ is not induced, $x_3^+v^{--} \in E(G_{i-1})$. However, we can now replace C^v, C^{x_2}, and C^{x_3} with $v^{--}x_3^+C^{x_3}x_3x_2^+C^{x_2}x_2v^-C^vv^{--}$, resulting in a 2-factor $\mathcal{C}'$ such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. This contradiction completes the proof of Case 3.5, and the proof of Theorem 5. $\square$

3 Disjoint 1-factors

As noted in the introduction, a 2-factor with no odd components is equivalent to a pair of disjoint 1-factors. Going forward we will refer to such a 2-factor as an *even 2-factor*. This section specifically considers Problem 1 when $k = 0$.

Problem 2. *Determine conditions in a graph G which guarantee that G has a pair of disjoint perfect matchings.*

In light of this problem, we have the following immediate corollary of Theorem 5 in the case $k = 0$.

Corollary 1. *Let G be a claw-free graph. Then G contains a pair of disjoint perfect matchings if and only if $cl(G)$ contains a pair of disjoint perfect matchings.*

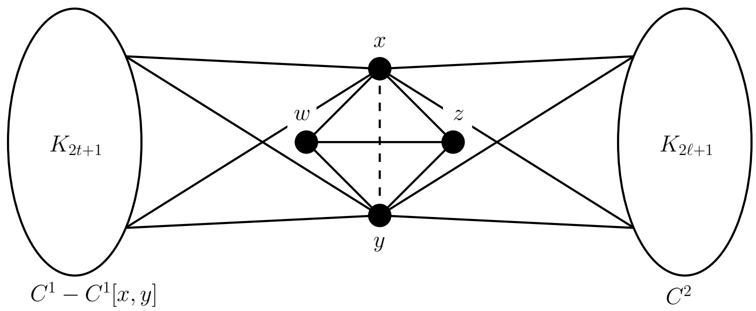

Figure 1: The family $\mathcal{E}$ of 2-connected $\{K_{1,4}, P_4\}$-free graphs that do not contain disjoint 1-factors.

It is clear that if a graph G has an even 2-factor, then G has a 2-factor, so we will consider conditions that ensure a graph has a 2-factor. In [10] Faudree, Faudree, and Ryjáček characterized the pairs of forbidden subgraphs which guarantee that a large enough 2-connected graph has a 2-factor.

Theorem 6. *Let X and Y be connected graphs with $X, Y \not\leq P_3$, and let G be a 2-connected graph of order $n \geq 10$. Then, G being $\{X, Y\}$-free implies that G has a 2-factor if and only if, up to the order of the pairs, $X = K_{1,3}$ and Y is a subgraph of either P_7, Z_4, $B(4,1)$, or $N(3,1,1)$, or $X = K_{1,4}$ and $Y = P_4$.*

We begin by considering $\{K_{1,4}, P_4\}$-free graphs. Let $\mathcal{E}$ be the family of graphs composed of two odd cliques and an edge wz joined to two vertices x and y where x may or may not be adjacent to y (see Figure 1). Every graph G in $\mathcal{E}$ is $\{K_{1,4}, P_4\}$-free and does not contain a pair of disjoint 1-factors, as the edge wz is in every perfect matching of G. Our next result demonstrates that graphs in the family $\mathcal{E}$ are the only $\{K_{1,4}, P_4\}$-free graphs with no even 2-factor. We require the following lemma from [10].

Lemma 1. *If G is a 2-connected $\{K_{1,4}, P_4\}$-free graph of order at least nine, then G has a 2-factor with at most two components.*

Theorem 7. *Any 2-connected $\{K_{1,4}, P_4\}$-free graph of even order at least ten contains an even 2-factor or is a member of the family $\mathcal{E}$.*

Proof. Let G be a 2-connected $\{K_{1,4}, P_4\}$-free graph of even order $n \geq 10$ that does not contain an even 2-factor. Thus every 2-factor has at least two components and by Lemma 1 there is a 2-factor $\mathcal{F}$ with exactly two components, call them C^1 and C^2. Since G is 2-connected there are at least two disjoint edges, e_1 and e_2, between C^1 and C^2. Let $e_1 = xx'$ and $e_2 = yy'$, where $x, y \in C^1$ and $x', y' \in C^2$. If $xy \in E(C^1)$ and $x'y' \in E(C^2)$, then C^1 and C^2 can be combined into a single cycle $xC^1yy'C^{2-}x'x$. Since G is P_4-free, $\langle x^-xx'x'^+\rangle$ is not an induced P_4. If $x^-x'^+$ is an edge, then $xC^1x^-x'^+C^2x'x$ is an even 2-factor in G, so without loss of generality, $xx'^+ \in E(G)$. Similarly, since $\langle x^-xx'^+x'^{++}\rangle$ is not an induced P_4, $xx'^{++} \in E(G)$. A similar argument shows that there is an induced P_4 for each edge from x to C^2 unless x dominates C^2. Now, if $xy \in E(C^1)$, then replace C^1 and C^2 with $yC^1xy'^+C^2y'y$, so $y^- \neq x$. For the same reason that x dominates C^2, y must also dominate C^2. (Note that if y' dominated C^1 instead, then we could combine C^1 and C^2.)

Let D be the set of vertices in C^1 that dominate C^2. There can be no two edges $e_1 = uv \in E(C^1)$ and $e_2 = u'v' \in E(C^2)$ such that $uu' \in E(G)$ and $vv' \in E(G)$, and there are at least two vertices in D, so $|V(C^1)| \geq 5$. If for some vertex $v \in D$, the edge v^-v^+ is in G, then C^1 can be shortened by using the edge v^-v^+ to skip v, and C^2 can be extended by including v, forming an even 2-factor. Thus no such edge exists. Since for any pair of vertices v_1 and v_2 in C^2, the graph $\langle x; x^-, x^+, v_1, v_2\rangle$ is not an induced $K_{1,4}$, the edge v_1v_2 exists. Thus $V(C^2)$ forms a clique.

Suppose e_1 and e_2 were chosen such that $C^1(x,y) \cap D = \emptyset$. Let v be any vertex in $C^1(x,y^-)$ such that xv is an edge of G. Then the edge xv^+ is in G since $\langle x'xvv^+\rangle$ is not an induced P_4. Now x dominates $C^1(x,y)$ and similarly y dominates $C^1(x,y)$. Let i be the smallest integer such that $x^{(+i)} \in C^1(x,y)$ does not dominate $C^1(x,y)$, and let j be the smallest integer such that $y^{(-j)}$ is not adjacent to $x^{(+i)}$. Consider $\left\langle x; x', x^-, x^{(+i)}, y^{(-j)}\right\rangle$. Either $x^-x^{(+i)}$ or $x^-y^{(-j)}$ is an edge, so that $x^-x^{(+i)}C^1y^-x^{(+i-1)}C^{1-}xx'C^2x'^-yC^1x^-$ or $x^-y^{(-j)}C^1y^-x^{(+i)}C^1y^{(-j-1)}x^{(+i-1)}C^{1-}xx'C^2x'^-yC^1x^-$ is a hamiltonian cycle. Consequently, no such i exists and $\left\langle C^1(x,y)\right\rangle$ forms a clique. If $|D| = 2$, then this argument can be repeated to show that $\left\langle C^1(y,x)\right\rangle$ is also a clique.

Suppose $|D| \geq 3$ and let $z \in D$ such that $C^1(x,z) \cap D = \{y\}$. As in the case that $|D| = 2$, $\left\langle C^1(y,z)\right\rangle$ forms a clique.

If there is an edge between $C^1(x,y)$ and $C^1(y,z)$, then we may rearrange C^1 such that y^-y^+ is an edge and remove y from C^1 and add it to C^2, and therefore no such edge exists. If $x^-y^+ \in E(G)$, then $C^1[x,y]$ can be pulled into C^2. Consider the path $\langle x^{--}x^-xx^+\rangle$. From before, x^-x^+ is not an edge, and $x^{--}x^+$ allows us to replace C^1 and C^2 with $x^{--}x^+C^{2-}xx^-C^1x^{--}$, so xx^{--} must be an edge. Iterating this argument, we get that x dominates C^1. However, since z dominates C^2, the cycle $xz^+C^1zx^+C^2x$ spans G and so is an even 2-factor. Thus, $|D| = 2$.

Now either $\langle xy^-yy^+\rangle$ or $\langle x^-xyy^+\rangle$ is an induced P_4 which is a contradiction implying that $|D| = 2$.

From before, $|C^1(x,y)| > 0$. Since C^1 is an odd cycle, either $|C^1(x,y)|$ or $|C^1(y,x)|$ is even, so without loss of generality, assume that $|C^1(x,y)|$ is even. If $|C^1(x,y)| \geq 4$, then $C^1[y,x)$, $C^1(x,y)$, and $C^2 + x$ form an even 2-factor. If $|C^1(x,y)| = 2$, then G is a member of $\mathcal{E}$. $\square$

Turning our attention to connected, but not necessarily 2-connected, graphs, Fujisawa and Saito [11] showed the following.

Theorem 8. *Let F_1 and F_2 be connected graphs of order at least three. Then there exists a positive integer n_0 such that every connected $\{F_1, F_2\}$-free graph of order at least n_0 and minimum degree at least two has a 2-factor if and only if $\{F_1, F_2\} \leq \{K_{1,3}, Z_2\}$.*

Theorem 9. *Every connected $\{K_{1,3}, Z_2\}$-free graph of even order with minimum degree $\delta(G) = \delta \geq 3$ contains an even 2-factor with at most two components.*

Proof. Let G be a connected $\{K_{1,3}, Z_2\}$-free graph of even order. Gould and Jacobson [13] showed that every 2-connected $\{K_{1,3}, Z_2\}$-free graph is hamiltonian, so we need only consider the case where G has a cut vertex x. Since G is $K_{1,3}$-free, x must lie in exactly two blocks, so let B_1 and B_2 be the blocks containing x.

Since G is claw-free, the neighborhood of any vertex is connected or two disjoint cliques, so $N_{B_1}(x)$ and $N_{B_2}(x)$ are cliques. Further, since $d(x) \geq 3$, without loss of generality, $|N_{B_1}(x)| \geq 2$. Let $v_1, v_2 \in N_{B_1}(x)$ and $u \in N_{B_2}(x)$. Suppose that $N_{B_2}(x) \neq V(B_2)$, and let $w \in V(B_2) \backslash N(x)$ such that $uw \in E(G)$. We know that there is such a w since G is connected. However, $\langle x, v_1, v_2; u, w \rangle$ is an induced Z_2. Thus $N_{B_2}(x) = V(B_2)$, and similarly $N_{B_1}(x) = V(B_1)$.

Finally, suppose there is some third block, B_3, which necessarily cannot contain x. Without loss of generality, let $v \in B_2 \cap B_3$ be a cut vertex of G. Let $u \in N_{B_3}(v)$, and that x is the only cut vertex of B_1. This means that since $\delta(G) \geq 3$, $|V(B_1)| \geq 4$ and x is adjacent to v. Now, let $x_1, x_2 \in V(B_1)$ such that $x_1 \neq x$ and $x_2 \neq x$. Then $\langle x_1, x_2, x; v, u \rangle$ is an induced Z_2, a contradiction.

Thus G consists of blocks B_1 and B_2, both of which are complete. Since G has even order, one block has even order and the other odd order. Without loss of generality, let B_1 have odd order. Then $B_1 - \{x\}$ and B_2 are even 2-connected $\{K_{1,3}, Z_2\}$-free graphs and so are hamiltonian, and this yields an even 2-factor of G. $\square$

Considering either a clique of any size with a pendant edge or the graph obtained by identifying a vertex in a clique of any size and a vertex of K_3, the minimum degree condition in our result cannot be weakened in order to still guarantee an even 2-factor in the graph.

3.1 Minimum Degree and Connectivity Conditions

There are a number of results that give minimum degree and connectivity conditions implying a claw-free graph has a 2-factor. Egawa and Ota [4] and Choudum and Paulraj [3] independently showed that a claw-free graph with minimum degree at least four has a 2-factor. Subsequently, Yoshimoto [32] and Aldred, Egawa, Fujisawa, Ota and Saito [1] demonstrated that $\delta(G) \geq 3$ suffices when G is 2-connected and claw-free.

We show next that there are no such conditions that ensure a 3-connected claw-free graph of even order contains an even 2-factor. Let $\mathcal{P}$ be the Petersen graph, and define $\mathbb{P}_\delta$ as the graph obtained by replacing each vertex of $\mathcal{P}$ with a clique of order $\delta + 1$. The three incident edges at one vertex v of $\mathcal{P}$ are now incident with three different vertices in the clique replacing v in $\mathbb{P}_\delta$. (See Figure 2 for $\mathbb{P}_4$.)

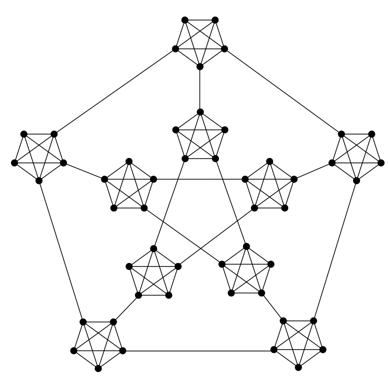

Figure 2: The graph $\mathbb{P}_4$.

Theorem 10. *For even $\delta \geq 4$, $\mathbb{P}_\delta$ is a 3-connected $K_{1,3}$-free graph with minimum degree δ and no even 2-factor.*

Proof. It is clear that $\mathbb{P}_\delta$ is 3-connected since $\mathcal{P}$ is 3-connected. The neighborhood of any vertex is either a clique or two disjoint cliques, thus $\mathbb{P}_\delta$ is $K_{1,3}$-free.

Consider a perfect matching M in $\mathbb{P}_\delta$, and a perfect matching M' that is disjoint from M. Let $F_{\mathcal{P}} \subset E(\mathbb{P}_\delta)$ be the set of edges that were originally edges of $\mathcal{P}$ (these are the edges that are not completely contained in a single clique of order $\delta + 1$). Each clique is odd, so M must contain $B \subseteq F_{\mathcal{P}}$ where each clique contains one or three vertices of $V(B)$. If $V(B)$ has three vertices in a single clique C, then there is no M', since $\mathbb{P}_\delta - M$ has C as an odd component. Therefore, $V(B)$ contains exactly one vertex in each clique, and corresponds to a perfect matching in $\mathcal{P}$. However, any perfect matching in $\mathcal{P}$ leaves two components, each isomorphic to C_5. The corresponding components in $\mathbb{P}_\delta$ are copies of C_5 with vertices replaced by odd cliques. These two components are odd (of order at least $5(\delta + 1)$), and so M' cannot exist. Thus, $\mathbb{P}_\delta$ does not contain an even 2-factor. $\square$

4 Open Problems

Theorem 10 implies that no minimum degree condition on 1-, 2-, and 3-connected claw-free graphs will ensure the existence of an even 2-factor. For 4-connected claw-free graphs, we look to the well-known conjecture of Matthews and Sumner [24].

Conjecture 1 (The Matthews Sumner Conjecture)**.** *If G is a 4-connected claw-free graph, then G is hamiltonian.*

A hamiltonian cycle in a graph of even order is an even 2-factor. We therefore make the following conjecture.

Conjecture 2. *If G is a 4-connected, claw-free graph of even order, then G has an even 2-factor if and only if G is hamiltonian.*

We also conjecture the following extension of Theorem 3, and in particular Corollary 1.

Conjecture 3. *If G is a k-connected, claw-free graph of even order at least $2k$, then G has k disjoint 1-factors if and only if $cl(G)$ has k disjoint 1-factors.*

For $k \geq 3$, the condition that G be k-connected in Conjecture 3 is necessary. To see this, consider the graph G obtained by connecting a vertex v to $k - 1$ vertices in an odd clique of order at least $2k - 1$. The closure of G is complete, but v does not have sufficient degree in G to be in k disjoint 1-factors.

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