PANCYCLICITY OF 4-CONNECTED, CLAW-FREE, P_{10}-FREE GRAPHS

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Abstract. A graph G is said to be pancyclic if G contains cycles of all lengths from 3 to $|V(G)|$. We show that if G is 4-connected, claw-free, and P_{10}-free, then G is either pancyclic or it is the line graph of the Petersen graph. This implies that every 4-connected, claw-free P_9-free graph is pancyclic, which is best possible and extends a result of Gould, Łuczak, and Pfender [R. Gould, T. Łuczak, and F. Pfender, Pancyclicity in 3-connected graphs: Pairs of forbidden subgraphs, *J. Graph Theory* **47** (2004), 183-202].

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1. Introduction and Notation

All graphs in this paper are simple. A graph G is *hamiltonian* if it contains a spanning cycle and *pancyclic* if it contains cycles of each length from 3 to $|V(G)|$. Throughout this paper, we will assume that all cycles C have an inherent clockwise orientation. For some vertex v on C we will denote the first, second, and i^{th} predecessor of v as v^-, v^{--}, and v^{-i} respectively. Similarly we denote the first, second, and i^{th} successor of v as v^+, v^{++}, and v^{+i} respectively. We also denote by $C(x,y)$ the set $\{x^+, x^{++}, \dots, y^-\}$ and by $C[x,y]$ the set $\{x, x^+, \dots, y\}$. We let xCy denote the path $xx^+ \dots y$ and xC^-y denote the path $xx^- \dots y$. Also, $xCyx$ denotes the cycle formed by adding an edge to the endpoints of the path xCy.

Given a family $\mathcal{F}$ of graphs, a graph G is said to be *$\mathcal{F}$-free* if G contains no member of $\mathcal{F}$ as an induced subgraph. If $\mathcal{F} = \{K_{1,3}\}$, then G is said to be *claw-free*. The following well-known conjecture of Matthews and Sumner [6] has provided the impetus for a great deal of research into the hamiltonicity of claw-free graphs.

Conjecture 1 (The Matthews-Sumner Conjecture)**.** *If G is a 4-connected claw-free graph, then G is hamiltonian.*

In [7], Ryjáček demonstrated that this is equivalent to a conjecture of Thomassen [12] that every 4-connected line graph is hamiltonian. Also in [7], Ryjáček showed that every 7-connected, claw-free graph is hamiltonian. Kaiser and Vrána [4] then showed that every 5-connected claw-free graph with minimum degree at least 6 is hamiltonian, which currently represents the best general progress towards affirming Conjecture 1. Recently, in [10], Conjecture 1 was also shown to be equivalent to the statement that every 4-connected claw-free graph is hamiltonian-connected.

The Matthews-Sumner Conjecture has also fostered a large body of research into other cycle-structural properties of claw-free graphs. In this paper, we are specifically interested in the pancyclicity of highly connected claw-free graphs. Significantly fewer results of this type can be found

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in the literature, in part because it has been shown in many cases [8, 9] that closure techniques such as those in [7] do not apply to pancyclicity.

In [11], Shepherd showed the following, which extended a well-known result of Duffus, Gould and Jacobson [1]. Here N denotes the *net*, which is a triangle with a pendant joined to each vertex.

Theorem 1 (Shepherd [11]). *Every 3-connected, $\{K_{1,3}, N\}$-free graph is pancyclic.*

Gould, Łuczak and Pfender [3] obtained the following characterization of forbidden pairs of subgraphs that imply pancyclicity in 3-connected graphs. Here Ł denotes the graph obtained by connecting two disjoint triangles with a single edge and $N(i, j, k)$ is the *generalized net* obtained by identifying an endpoint of each of the paths P_{i+1}, P_{j+1} and P_{k+1} with distinct vertices of a triangle.

Theorem 2 (Gould, Łuczak, Pfender [3]). *Let X and Y be connected graphs on at least three vertices. If neither X nor Y is P_3 and Y is not $K_{1,3}$, then every 3-connected $\{X, Y\}$-free graph G is pancyclic if, and only if, $X = K_{1,3}$ and Y is a subgraph of one of the graphs in the family*

$$\mathcal{F} = \{P_7, Ł, N(4, 0, 0), N(3, 1, 0), N(2, 2, 0), N(2, 1, 1)\}.$$

Motivated by the Matthews-Sumner Conjecture and Theorem 2, Ron Gould posed the following problem at the 2010 SIAM Discrete Math meeting in Austin, TX.

Problem 1. *Characterize the pairs of forbidden subgraphs that imply a 4-connected graph is pancyclic.*

Observe that in Theorem 2, all generalized nets of the form $N(i, j, 0)$ with $i + j = 4$ are in the family $\mathcal{F}$. With this in mind, Ferrara, Gould, Gehrke, Magnant, and Powell [2] showed the following. The line graph of the Petersen graph is 4-connected, claw-free and contains no cycle of length 4 (see Figure 1).

Theorem 3 (Ferrara, Gould, Gehrke, Magnant, Powell [2]). *Every 4-connected $\{K_{1,3}, N(i, j, 0)\}$-free graph with $i + j = 6$ is pancyclic. This result is best possible, in that the line graph of the Petersen graph is $N(i, j, 0)$-free for all $i, j \geq 0$ with $i + j = 7$.*

The main result of this paper is as follows.

Theorem 4. *Every 4-connected, claw-free, P_{10}-free graph is either pancyclic or is the line graph of the Petersen graph.*

This immediately implies the following corollary, which is best possible in light of Theorems 1 and 4, and represents new progress towards Problem 1.

Theorem 5. *Every 4-connected, claw-free, P_9-free graph is pancyclic.*

2. Long Cycles

Let G be an n-vertex, 4-connected, claw-free, P_{10}-free graph. We will first show that G contains cycle of all lengths from 10 up to n, following closely the technique used by Ferrara, Gould, Gehrke, Magnant, and Powell in [2]. To this end, the following lemma of Gould, Łuczak, and Pfender [3] will be valuable. A *hop* is a chord in a cycle that joins some vertex v to v^{++}.

Lemma 1 (Gould, Łuczak, Pfender [3]). *Let G be a claw-free graph with minimum degree at least 3, let C be a cycle of length $t \geq 5$ without hops, and let X be the set of vertices in C that are not on any chord of C. If some chord xy of C satisfies $|X \cap C(x, y)| \leq 2$, then G contains cycles of lengths $t - 1$ and $t - 2$.*

We use Lemma 1 to prove the following lemma.

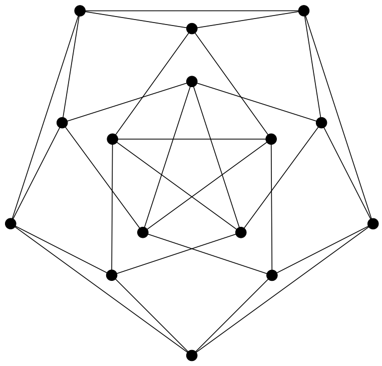

FIGURE 1. The line graph of the Petersen graph is the unique 4-connected, claw-free, P_{10}-free graph that is not pancyclic.

Lemma 2. *Let G be a 4-connected, claw-free graph of order n and suppose that G contains a cycle of length $t \geq 5$ with at least one chord. If G contains no cycle of length $t-1$ then G contains an induced copy of P_{10}.*

Proof. Let C be a cycle of length t in G with at least one chord and suppose that G contains no $(t-1)$-cycle. Let X be the set of vertices of C that are not endpoints of chords of C. Let xy be a chord of C such that $|C(x,y) \cap X|$ is minimized and xy is the only chord that joins a pair of vertices in $C[x,y]$. Furthermore, choose xy so that $x^+y^+ \notin E(G)$. Since G contains no $(t-1)$-cycle, C is hop-free and by Lemma 1, $|C(x,y) \cap X| \geq 3$. We begin by showing that $|C(x,y)| \geq 5$.

Assume that $|C(x,y)| = 3$. By minimality of xy, there are at least three vertices in X contained in $C(y,x)$. Let x_1, x_2, x_3 be three such vertices labeled in order. Since G is 4-connected, each vertex in X has at least two neighbors not on C. If a vertex v in C has a neighbor v' not in C, then v' is also adjacent to v^- or v^+ as G is claw-free and C is hop-free. A *claw-extension* of C is the extension of C by replacing v^-vv^+ with $v^-vv'v^+$ or $v^-v'vv^+$. Thus the $(t-3)$-cycle $xyCx$ can be extended to a $(t-1)$-cycle via claw-extensions at x_1 and x_3.

Now assume that $|C(x,y)| = 4$. We show that all four vertices in $C(x,y)$ are also in X. Suppose one of them is not; call it p. By minimality of xy, we conclude that p has a neighbor q in $C(y,x)$. Pick q to have minimum distance from y^+ along C^+. Because G is claw-free and C is hop-free, $pq^+ \in E(G)$. Let x_1 be a vertex in $C(y,x) \cap X$; note that x_1 exists because $|C(y,x) \cap X| \geq 3$ by assumption, and also x_1 is not q or q^+. Let u and v be two neighbors of x_1 that are not in C. Both u and v have a neighbor in $\{x_1^+, x_1^-\}$. If u and v are both adjacent to x_1^+ and not adjacent to x_1^- (similarly, adjacent to x_1^- and not adjacent to x_1^+), then u and v are adjacent. Otherwise, it is possible to pick distinct neighbors for x_1^+ and x_1^- from $\{u, v\}$; without loss of generality we assume that $ux_1^+, vx_1^- \in E(G)$. Thus it is possible to replace $C[x_1^-, x_1^+]$ with $x_1^-uvx_1x_1^+$, $x_1^-x_1uvx_1^+$, or $x_1^-vx_1ux_1^+$, and replace qq^+ with qpq^+, thereby extending the $(t-4)$-cycle $xyCx$ to a $(t-1)$-cycle. Thus every vertex in $C(x,y)$ is also in X.

By the minimality of xy, there are at least four vertices in $C(y,x) \cap X$; let $X' = \{x_1, x_2, x_3, x_4\}$ be a set of four such vertices labeled with respect to the orientation of C. We have shown that it is possible to add two vertices not in C to $C[x_i^- x_i x_i^+]$ for any $i \in [4]$. Thus, if $|N(X') - C| \geq 3$, then we can extend the $(t-4)$-cycle $xyCx$ to a $(t-1)$-cycle. Hence we assume that there are exactly two vertices, u and v, in $N(X') - C$ and that each vertex in X' is adjacent to both u and v. If any three vertices in X' are pairwise non-consecutive on C, then there is a claw centered at u. Also, if there are exactly two vertices in C between x_i and x_j in X', then G contains the $(t-1)$-cycle $x_iux_jCx_i$ (assuming $i < j$). Thus $x_1x_2, x_3x_4 \in E(C)$, but $x_2x_3 \notin E(C)$ and there are at least three vertices between x_2 and x_3. If u (similarly v) is adjacent to x_2^+ then G contains the claw $\langle u; x_1, x_2^+, x_4 \rangle$. We conclude that $uv \in E(G)$, otherwise G contains the claw $\langle x_2; x_2^+, u, v\rangle$. Because x_2^+ is not adjacent to u, we may assume that x_2^+ is an endpoint of a chord on C. Let x_2' be the other endpoint of this chord. Since each vertex in $C(x,y)$ is in X, it follows that $x_2' \in C[y,x]$. By minimality of xy, both $C(x_2^+, x_2')$ and $C(x_2', x_2^+)$ contain at least four vertices in X. Thus $|C(y,x) \cap X| \geq 6$ and all six of these vertices are adjacent to both u and v. It follows that there is a claw centered at u, a contradiction. Thus $|C(x,y)| \neq 4$, and we conclude that $|C(x,y)| \geq 5$.

Let $k = |C(x,y)|$. Recalling that we chose xy so that $x^+y^+ \notin E(G)$, we observe that, as G is claw-free and hop-free, x^-y, xy^+, and x^-y^+ are edges. Observe that $x^+Cyx^-C^-x^{-(9-k)}$ is a 10-vertex path. If there is no chord other than yx^- joining $C[x^+, y]$ and $C[x^{-(9-k)}, x^-]$, then the path is induced. Let $w \in C[x^{-(9-k)}, x^-]$ and $z \in C[x^+, y]$ be neighbors. If $z \in \{y^-, y\}$, then G contains a cycle on $(t-1)$ or $(t-2)$ vertices in $V(C)$ that contains the path xCy^-. If necessary, such a cycle can be extended to a $(t-1)$-cycle via a claw-extension. Thus $z \in C[x^+, y^{--}]$. The combination of the minimality of xy and the fact that $z \neq y^-$ then implies that $w \neq x^-$.

We now choose wz so that $|C(w,z)|$ is minimized. The minimality of wz implies that w^-z and wz^+ are edges. Let $A = C(x,z)$, $B = C(z^+, y)$, and $D = C(w, x^-)$. Note that $|D| \leq 2$, $|A| + |B| = k - 2$, and $|A| + |B| + |D| \leq 5$. Consider the two cycles $C' = wz^+C^-x^-yCw$ and $C'' = wzCyxx^-y^+Cw$. Observe that C' contains no vertices in $B \cup D$ and C'' contains no vertices in $A \cup D$, thus neither of these cycles contains t vertices.

If $|D| = 2$, then $k = 5$ and $A \cup B \subset X$. Furthermore, either $|A| = 1$ or $|B| = 1$ and either C' or C'' is a $(t-3)$-cycle. Such a cycle can be extended to a $(t-1)$-cycle using claw-extensions. If $|D| \leq 1$, then by the minimality of xy there are at least three vertices in $C(y^+, x^-) \cap X$, at least two of which lie in $C(y^+, w)$. Thus if C' or C'' has length at least $(t-3)$ we can extend to a $(t-1)$-cycle using claw-extensions. Therefore G contains a $(t-1)$-cycle unless $|A| + |D| \geq 4$ and $|B| + |D| \geq 4$, which together with the fact that $|A| + |B| + |D| \leq 5$ yields a contradiction. □

The following result of Łuczak and Pfender [5] allows us to obtain the desired long cycles.

Theorem 6 (Łuczak, Pfender [5]). *Every 4-connected, claw-free, P_{11}-free graph is hamiltonian.*

If follows that every 4-connected, claw-free, P_{10}-free graph G is hamiltonian. The assumption that G is P_{10}-free implies that G contains no induced cycle of length at least 11. Therefore, by Lemmas 1 and 2, G contains cycles of length $|V(G)|$ through 10.

3. Short Cycles

In this section we show that a 4-connected, claw-free, P_{10}-free graph G with at least nine vertices either contains cycles of lengths 3 through 9 or is the line graph of the Petersen graph. The existence of a 3-cycle follows immediately from the fact that G is claw-free. For 4- and 5-cycles we use similar arguments based on longest induced paths. For 6-, 7-, and 8-cycles we use an argument based on the neighborhoods of vertices in smaller cycles. Finally, the existence of a 9-cycle follows from the existence of 10-cycles.

We begin by showing that G either contains a 4-cycle or is the line graph of the Petersen graph. Let Z_t be the graph obtained from the disjoint union of P_{t+1} and K_3 by identifying one end-vertex of P_{t+1} with one vertex in K_3. That is $Z_t \cong N(t,0,0)$. We use the following, which is an immediate consequence of Theorem 3.

Theorem 7. *Every 4-connected, claw-free, Z_5-free graph is pancyclic.*

Lemma 3. *If G is a 4-connected, claw-free, P_{10}-free graph, then either G is the line graph of the Petersen graph or G contains a 4-cycle.*

Proof. Suppose that G does not contain a 4-cycle. Since G is claw-free, the neighborhood of any vertex is either connected or two cliques. As G has minimum degree 4, if the neighborhood is connected, then the neighborhood contains a path of order 3, yielding a 4-cycle. Thus the neighborhood of any vertex is two cliques. If any vertex has degree at least 5, then one of the cliques has at least three vertices, yielding a 4-cycle. Thus G is 4-regular and the closed neighborhood of each vertex induces two triangles identified at that vertex.

We consider a longest induced path P in G; let $v_1, \dots, v_t$ be the vertices of P labeled in order. We have shown that v_1 has exactly three neighbors that are not in P, call these vertices x_1, x_2, and x_3. Because v_1 lies in two triangles, we may assume without loss of generality that x_3 is adjacent to v_2 and that x_1 and x_2 are adjacent. Because G does not contain P_{t+1} as an induced subgraph, both x_1 and x_2 must have neighbors in $V(P) - v_1$. Since G does not contain a 4-cycle, x_1 and x_2 have no common neighbors besides v_1 and are not adjacent to v_2 or v_3. Furthermore, a neighbor of x_1 cannot be adjacent to a neighbor of x_2. Also, because G is claw-free, any vertex that is adjacent to v_i for $i \in \{2, \dots, t-1\}$ must also be adjacent to v_{i+1} or v_{i-1}.

By Theorem 7, we may assume that G contains an induced copy of Z_5, and therefore $t \geq 7$. If $t = 7$, then (up to symmetry) x_1 is adjacent to v_4 and v_5, and x_2 is adjacent to v_7. It follows that there is a vertex x_2' that is adjacent to x_2 and v_7, which must also have neighbors in $V(P) - v_7$. The only possible neighbor of x_2' that does not complete a 4-cycle is v_3, and thus G contains either a 4-cycle or an induced claw.

Now assume that $t = 8$; there are three cases to consider. If x_1 is adjacent to v_4 and v_5 and x_2 is adjacent to v_7 and v_8, then v_8 has two additional neighbors y_1 and y_2, both of which have neighbors in $V(P) - v_8$. Because G is claw-free and has no 4-cycle, these neighbors lie in the set $\{v_4, v_3, v_2\}$; thus y_1 and y_2 have a second common neighbor, yielding a 4-cycle. If x_1 is adjacent to v_4 and v_5 and x_2 is adjacent v_8 but not v_7, then there is a vertex x_2' that is adjacent to x_2 and v_8. If x_2' has no neighbor in $V(P) - v_8$, then G contains the induced 9-vertex path $x_2'x_2v_1 \dots v_7$; as before, the only possible neighbor of x_2' that does not complete a 4-cycle is v_3, and thus G contains a 4-cycle or an induced claw. Finally, suppose that x_1 is adjacent to v_5 and v_6 and x_2 is adjacent to v_8. If x_3 does not have additional neighbors in P, then G contains the induced path $x_2v_8 \dots v_2x_3$. The only possible neighbor of x_3 in $V(P) - v_1$ that does not complete a 4-cycle is v_7, and thus G contains a 4-cycle or an induced claw. We conclude that the longest induced path in G contains nine vertices.

Both v_1 and v_9 have three neighbors that do not lie in P. Let y_1, y_2, and y_3 be the neighbors of v_9 with y_3 adjacent to v_8; note that they are not necessarily distinct from x_1, x_2, and x_3 (the neighbors of v_1, as before). Suppose that x_1, x_2, x_3, y_1, y_2, and y_3 are all distinct. Since x_1 and x_2 both have neighbors in P, we conclude (up to symmetry) that x_1 is adjacent to v_4 and v_5 and x_2 is adjacent to v_7 and v_8. Likewise we may assume that y_1 is adjacent to v_5 and v_6 and that y_2 is adjacent to v_2 and v_3. Because G is claw-free and contains no 4-cycle, x_3 has no additional neighbors in P. Thus $x_3v_2v_3v_4x_1x_2v_7v_6y_1v_9$ is an induced 10-vertex path. Therefore v_1 and v_9 have a common neighbor and since G contains no 4-cycle, we conclude that they have exactly one common neighbor.

Suppose first that $x_3 = y_3$; that is, there is a vertex that is adjacent to v_1, v_2, v_8, and v_9. Thus x_1 and x_2 cannot be adjacent to v_1, v_2, v_3, v_8, or v_9, and therefore have neighbors in $\{v_4, \dots, v_7\}$. It then follows that G contains a 4-cycle or an induced claw. Now suppose that $x_2 = y_2$; that is, there is a vertex that is adjacent to v_1 and v_9 but no other vertices in P. It follows that the neighbors of x_1 are a pair of consecutive vertices in $\{v_4, \dots, v_7\}$ and the neighbors of y_1 are a pair of consecutive vertices in $\{v_3, \dots, v_6\}$. Since x_1 and y_1 cannot have a common neighbor in P (such a vertex completes a 4-cycle), we conclude that x_1 is adjacent to v_6 and v_7 or that y_1 is adjacent to v_3 and v_4. Assuming, without loss of generality, that x_1 is adjacent to v_6 and v_7, it follows that x_3 has no neighbors in $P - \{v_1, v_2\}$. Thus $x_2 v_9 \dots v_2 x_3$ is an induced 10-vertex path.

It remains to consider the case when (up to symmetry) $x_3 = y_2$; that is, there is a vertex that is adjacent to v_1, v_2, and v_9 (see Figure 2). The neighbors of x_1 and x_2 are pairs of consecutive vertices from $\{v_4, \dots, v_8\}$, and since x_1 and x_2 cannot have adjacent neighbors, we may assume (up to symmetry) that x_1 is adjacent to v_4 and v_5, and x_2 is adjacent to v_7 and v_8. In this case, since the neighbors of y_1 are consecutive vertices in $\{v_4, v_5, v_6\}$, we conclude that y_1 is adjacent to v_5 and v_6. We consider the neighbors of y_3. If y_3 has two neighbors that are not in P, then each must have neighbors in P to avoid an induced 10-vertex path. The neighbors of these vertices must be consecutive vertices from the set $\{v_2, v_3, v_4, v_6, v_7\}$ (these are the vertices that do not yet have four neighbors). If one is adjacent to v_6 and v_7, then G contains a 4-cycle. Otherwise, they are both adjacent to v_3, which again yields a 4-cycle. We conclude that all neighbors of y_3 are in P. Since G does not contain 4-cycles and the neighbors of y_3 lie in $\{v_2, v_3, v_4, v_6, v_7\}$, we conclude that y_3 is adjacent to v_3 and v_4. Finally we consider the remaining neighbors of v_2, v_3, v_6, and v_7. Since G is claw-free, v_2 and v_3 have a common neighbor u. If u is not adjacent to v_6 and v_7, then $v_1 x_3 v_9 \dots v_3 u$ is an induced 10-vertex path. Thus u is also adjacent to v_6, and v_7. As G is 4-regular, there are no remaining vertices and G is the line graph of the Petersen graph. $\square$

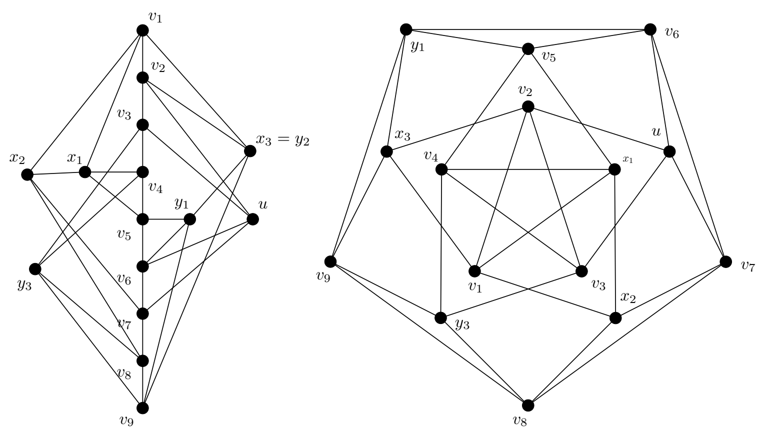

FIGURE 2. Two drawings of the line graph of the Petersen graph.

To show the existence of 5-cycles we use an argument similar to the proof of Lemma 3. Note that the line graph of the Petersen graph does contain a 5-cycle; thus the following proof applies to all 4-connected, claw-free, P_{10}-free graphs.

Lemma 4. *If G is a 4-connected, claw-free, P_{10}-free graph, then G contains a 5-cycle.*

Proof. Suppose that G does not contain a 5-cycle. If G contains a 4-cycle, then each vertex in the cycle has a neighbor not in the cycle since G is 4-connected. Because G is claw-free and has no 5-cycle, the vertex set of the 4-cycle is a clique. Let $\{v_1, \dots, v_t\}$ be the vertex set of a longest induced path P in G; by Theorem 7, $t \geq 7$. We first consider the case when v_1 has two neighbors x_1 and x_2 that are adjacent such that x_2 is also adjacent to v_2 (see Figure 3).

Since $\{v_1, v_2, x_1, x_2\}$ is the vertex set of a 4-cycle, it is a clique. Because G is 4-connected, v_1 has a third neighbor not in P; call it x_3. As P is a longest induced path and G contains no 5-cycle, x_3 must have a neighbor in $\{v_5, \dots, v_t\}$. Observe that x_3 and x_1 (similarly x_2) cannot have neighbors in P that are common or adjacent. Since a neighbor of an internal vertex in P must be adjacent to consecutive vertices in P, we conclude that x_1 and x_2 do not have any neighbors in $V(P) - \{v_1, v_2\}$ when $t = 7$. Furthermore, when $t \in \{8, 9\}$, at most one of x_1 and x_2 can

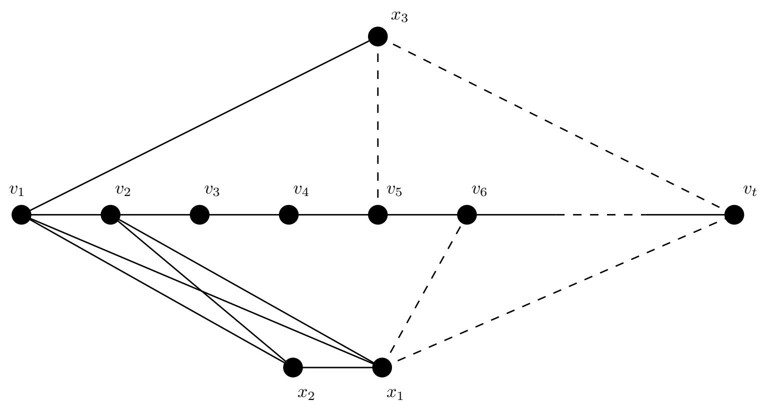

FIGURE 3. The case when v_1 has two adjacent neighbors x_1 and x_2 with x_2 adjacent to v_2.

have neighbors in $V(P) - \{v_1, v_2\}$. Without loss of generality, assume that x_1 has neighbors in $\{v_3, \ldots, v_t\}$. Because G does not contain 5-cycles, if x_1 is adjacent to v_i, then $i \geq 6$. It follows that there are five possible ways that x_1 and x_3 can have neighbors in $V(P) - \{v_1, v_2\}$ without creating induced claws or 5-cycles:

$t = 8$ and $x_3v_5, x_3v_6, x_1v_8 \in E(G)$;

$t = 9$ and $x_3v_5, x_3v_6, x_1v_9 \in E(G)$;

$t = 9$ and $x_3v_5, x_3v_6, x_1v_8, x_1v_9 \in E(G)$;

$t = 9$ and $x_3v_6, x_3v_7, x_1v_9 \in E(G)$;

$t = 9$ and $x_3v_9, x_1v_6, x_1v_7 \in E(G)$.

In all five cases, consider a neighbor x_2' of x_2, which exists because G is 4-connected. If x_2' has any neighbors in P, then G contains either a 5-cycle or an induced claw. Therefore $v_t \ldots v_2, x_2, x_2'$ is an induced path on $t+1$ vertices. We conclude that neither x_1 nor x_2 has neighbors in $V(P) - \{v_1, v_2\}$.

Because G is 4-connected, x_1 and x_2 have distinct neighbors x_1' and x_2' that are not in $V(P) \cup \{x_1, x_2, x_3\}$. It is clear that $\{x_1', x_2', x_3\}$ is an independent set, otherwise G contains a 5-cycle. Because P is an induced path with the maximum number of vertices, x_1', x_2' and x_3 all must have neighbors in P. Furthermore, these neighbors must be distinct. It follows that this is only possible if $t = 9$ and (up to symmetry), x_1' is adjacent to v_5 and v_6, x_2' is adjacent to v_7 and v_8, and x_3 is adjacent to v_9. However, $x_3v_9v_8v_7v_6x_1'x_1v_2v_3v_4$ is then an induced 10-vertex path. We conclude that v_1 does not have two adjacent neighbors that are adjacent to v_2.

We now consider the case when the common neighbors of v_1 and v_2 are an independent set. Since v_1 has at least three neighbors that are not in P, each of which has a neighbor in $P - v_1$, it is clear that v_1 and v_2 must have a common neighbor; call it x_3. Furthermore, since G is claw-free, v_1 has two neighbors not in P that are adjacent; again we call them x_1 and x_2. Both x_1 and x_2 must have neighbors in $P - v_1$, and these neighbors cannot be in $\{v_2, v_3, v_4\}$, nor can they be adjacent or at distance 2 in P. Thus there are two cases to consider. First suppose that x_1 and x_2 are adjacent to v_t. Because G is 4-connected and claw-free x_1 has an additional neighbor x_1' that is adjacent to v_1 or v_t completing a 5-cycle. In the second case, $t = 9$ and, up to symmetry, $x_1v_5, x_1v_6, x_2v_9 \in E(G)$. Note that in this case, x_3 cannot have any additional neighbors in P. In particular, x_3 is not adjacent to v_3 because the 4-cycle on $\{x_3, v_3, v_2, v_1\}$ would force a clique, and P would not be induced. Thus x_3 has at least two additional neighbors x_3' and x_3'' that are not in P and which, by assumption, are not adjacent to v_2. The only possible neighbors that these vertices can have in P are v_7 and v_8, and because G is claw-free and has no 5-cycle we may assume that x_3' does not have any neighbors in P. Therefore $v_9 \dots v_2 x_3 x_3'$ is an induced 10-vertex path. $\square$

The following notation is useful for the proof of the next lemma. For a set of vertices S, let $N^k(S) = \{v | \min\{d(v, v_i)\} = k,\ v_i \in S\}$. Also, $G[S]$ denotes the subgraph of G induced by S.

Lemma 5. *If G is a 4-connected, claw-free, P_{10}-free graph, then G contains cycles of length 6, 7, and 8.*

Proof. By Lemma 4, G contains a 5-cycle. Let t be the largest integer less than 8 such that G contains a t-cycle but no $(t+1)$-cycle, and let C be a t-cycle in G. Since G is 4-connected, C has at least four vertices v_1, v_2, v_3, and v_4 with a neighbor not in C. If v_i has a neighbor w_i that is not in $V(C)$ such that w_i is adjacent to either v_i^+ or v_i^-, then G contains a $(t+1)$-cycle. Thus, since G is claw-free, we may assume that v_i^+ is adjacent to v_i^-. Using similar arguments we conclude that $G[V(C)]$ contains one of the graphs in Figure 4 as a subgraph, where v_1, v_2, v_3, and v_4 are the vertices incident to the dashed edges.

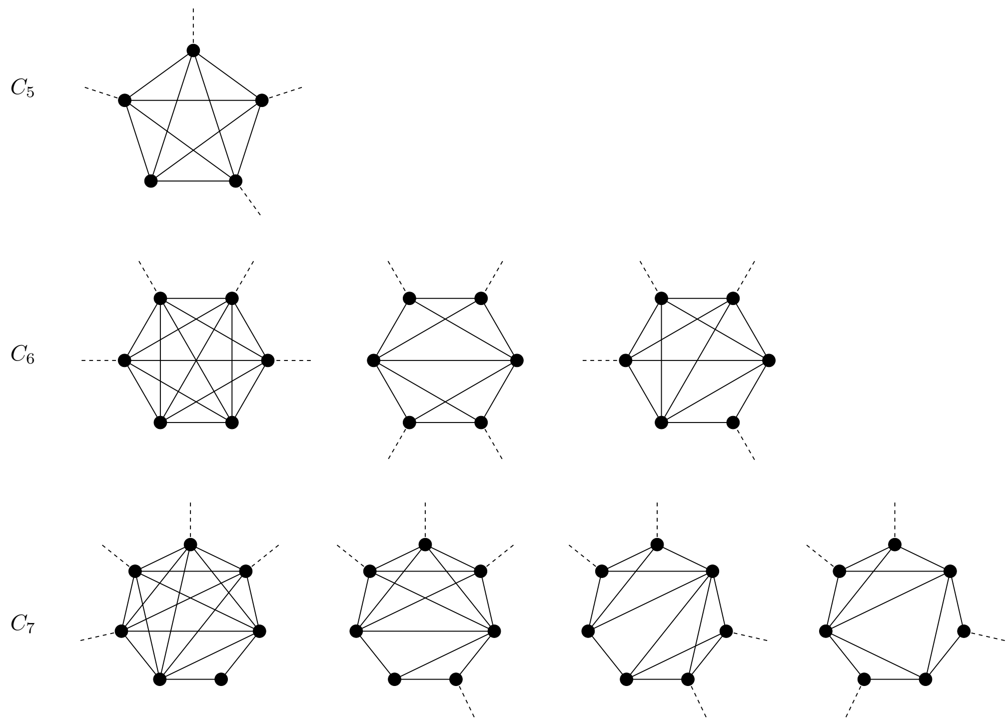

FIGURE 4. Necessary subgraphs in $G[V(C)]$ with v_1, v_2, v_3, and v_4 incident to the dashed edges.

If $t = 5$, then $G[V(C)]$ contains paths of length 1 through 4 joining each pair of vertices taken from $\{v_1, \dots, v_4\}$. Similarly, if $t \geq 6$, then $G[V(C)]$ contains paths of length 2 through $t-1$ joining each pair of vertices taken from $\{v_1, \dots, v_4\}$. Let $P_{(i,j)}$ be a shortest path in $G[V(C)]$ connecting v_i and v_j that does not contain v_k for any k distinct from i and j. For $i \in \{1, 2, 3, 4\}$, let $V_i = N(v_i) \cap N^1(V(C))$, and let $V_i' = N^2(v_i) \cap N^2(V(C))$. We conclude that for $1 \leq i < j \leq 4$, the following hold:

(a) $V_i \cap V_j = \emptyset$ for $i \neq j$;

(b) For $i \neq j$, there are no edges joining V_i and V_j;

(c) $V_i' \neq \emptyset$ for $i \in [4]$;

(d) $V_i' \cap V_j' = \emptyset$ for $i \neq j$;

(e) For $i \neq j$, there are no edges joining V_i' and V_j'.

The structure of $G[V(C)]$ and the assumption that G does not contain a $(t+1)$-cycle imply (a), (b), (d), and (e). Since G is 4-connected, (c) must hold, as otherwise v_i is a cut-vertex. Thus G has the structure shown in Figure 5.

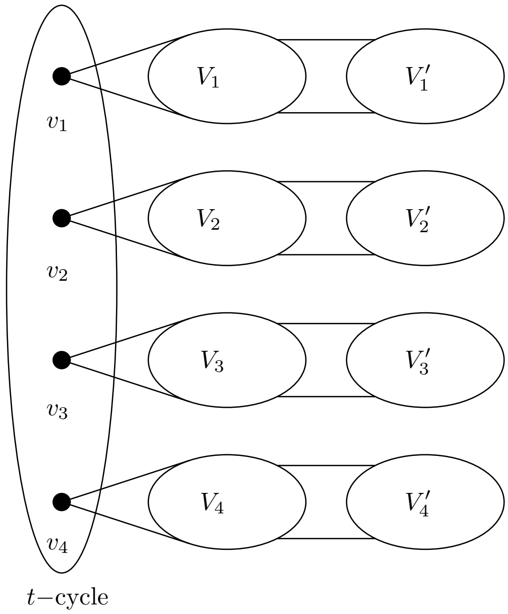

FIGURE 5. The structure of G.

Given disjoint sets $S_1, \ldots, S_\ell \subset V(G)$ and $u, v \in V(G) - (S_1 \cup \ldots \cup S_\ell)$, we let $uS_1 \ldots S_\ell v$ denote a general $u - v$ path of the form $us_1 \ldots s_\ell v$, where $s_i \in S_i$ for $i \in [\ell]$.

Let $V_i'' = N^3(v_i) \cap N^3(V(C))$. We now show that $V_i'' \cap V_j'' = \emptyset$ for $i \neq j$. Using (e) and the fact that G is claw-free, we see that no vertex can have a neighbor in V_i' for three distinct values of i. Suppose first that there are vertices $x_{(1,2)} \in V_1'' \cap V_2''$ and $x_{(3,4)} \in V_3'' \cap V_4''$. Since G is claw-free, $x_{(1,2)}x_{(3,4)} \notin E(G)$, implying that $V_1V_1'x_{(1,2)}V_2'V_2v_2P_{(2,3)}v_3V_3V_3'x_{(3,4)}$ is an induced path on at least 10 vertices.

Now suppose that there are vertices $x_{(1,2)} \in V_1'' \cap V_2''$ and $x_{(1,3)} \in V_1'' \cap V_3''$. Suppose that $x_{(1,2)}$ and $x_{(1,3)}$ have a common neighbor y in X_1''. Since G is claw-free, $x_{(1,2)}x_{(1,3)} \in E(G)$ and $V_2V_2'x_{(1,2)}x_{(1,3)}V_3'V_3v_3P_{(3,4)}v_4V_4V_4'$ is an induced path on at least 10 vertices. Thus we may assume that $x_{(1,2)}$ has a neighbor y in V_1'' that is not adjacent to $x_{(1,3)}$ and that $x_{(1,2)}x_{(1,3)} \notin E(G)$. It follows that $V_2V_2'x_{(1,2)}yV_1v_1P_{(1,3)}v_3V_3V_3'x_{(1,3)}$ is an induced path on at least 10 vertices. Similarly, if V_3'' is nonempty and $V_3'' \cap V_i'' = \emptyset$ for $i \in \{1,2,4\}$ then $V_1V_1'x_{(1,2)}V_2'V_2v_2P_{(2,3)}v_3V_3V_3'V_3''$ is an induced path containing at least 10 vertices.

By symmetry, we conclude that $V_i'' \cap V_j'' = \emptyset$ when $1 \leq i < j \leq 4$. Since G is 4-connected there is a path from each V_i'' to each V_j'' when $i \neq j$ that contains no vertices in C. Let P' be a shortest such path connecting V_i'' and V_j'' over all choices of i and j; we assume without loss of generality that $i = 1$ and $j = 2$. Since P' is minimal, $V(P') \cap V_k = \emptyset$ and $V(P') \cap V_k' = \emptyset$ for all $k \in [4]$. Thus $V_1V_1'V_1''P'V_2''V_2'V_2v_2P_{(2,3)}v_3V_3V_3'$ is an induced path on at least 10 vertices.

Therefore, G contains a $(t+1)$-cycle. $\square$

We now show that 4-connected, claw-free, P_{10}-free graphs contain 9-cycles, completing the proof of Theorem 4.

Lemma 6. *If G is a 4-connected, claw-free, P_{10}-free graph, then G contains a 9-cycle.*

Proof. Suppose that G does not contain a 9-cycle. By Lemma 2, G contains a 10-cycle C, and we let $\{v_1, \dots, v_{10}\}$ be the vertex set of C labeled in order. Also by Lemma 2, we may assume that C is chordless. Let v' be a neighbor of v_1 not in $V(C)$. Since G is claw-free, we may assume, without loss of generality, that v' is also adjacent to v_{10}. If $|N(v') \cap V(C)| = 2$, then $v'v_1Cv_9$ is an induced path on 10 vertices. If v' is at adjacent to v_i for $i \in \{3, 4, 7, 8\}$, then G contains a 9-cycle. We consider the case when v' is adjacent to v_5 and, as G is claw-free, v_6. Because G is 4-connected, v_3 has a neighbor v'' that does not lie in C; as G is claw-free, v'' is also adjacent to v_2 or v_4. It follows that $\{v_{10}, v_1, \dots, v_6, v', v''\}$ is the vertex set of a 9-cycle. Thus, we may assume that v' is adjacent to v_2 or v_9; if v' is adjacent to both, then $v_9v'v_2v_3 \dots v_9$ is a 9-cycle. We conclude that every vertex with a neighbor on C has exactly three neighbors on C which are consecutive.

For $1 \leq i \leq 10$, let $V_i = N(v_{i-1}) \cap N(v_i) \cap N(v_{i+1})$ where indices are taken modulo 10. If there is a vertex $w \notin V(C) \cup \bigcup_{i \in [10]} V_i$ that has a neighbor w_i in some V_i, then $\langle w_i; v_{i-1}, v_{i+1}, w \rangle$ is a claw. Thus we may assume that the sets $V_1, \dots, V_{10}$ partition $V(G) \setminus V(C)$. If there is an edge joining V_i and V_j when $|i-j| \geq 2 \pmod{10}$, then G contains a 9-cycle. If there are two nonconsecutive values $i < j$ such that V_i and V_j are empty, then $\{v_i, v_j\}$ is a cutset, a contradiction. Thus for some $1 \leq i \leq 10$, the sets V_i, V_{i+1}, V_{i+2}, and V_{i+3} are all non-empty. Let w_j be any vertex in V_j for $i \leq j \leq i+3$. It follows that $v_iw_iv_{i+1}w_{i+2}v_{i+3}v_{i+4}w_{i+3}v_{i+2}w_{i+1}v_i$ is a 9-cycle. $\square$

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